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THE ONE-QUARTER THEOREM FOR MEAN UNIVALENT FUNCTIONS

BY

P. R. GARABEDIAN AND H. L. ROYDEN

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THE ONE-QUARTER THEOREM FOR MEAN UNIVALENT FUNCTIONS

by

P. R. Garabedian and H. L. Royden

1. Formulation and history of the problem.

Many attempts have been made to generalize the fundamental distortion theorems of the theory of schlicht functions to the case of p-valued functions. The derivation of sharp bounds for p-valent functions turns out to be none too easy. Many of the estimates which have been found, however, are valid for wider classes of functions satisfying only a condition of mean p-valence.

In particular, Spencer [5] has studied the class of mapping functions

$$w = f(z) = z^p + a_{p+1}z^{p+1} + a_{p+2}z^{p+2} + \dots \quad , \tag{1}$$

regular in the unit circle $|z| < 1$, which transform the unit circle into a Riemann surface R over the w-plane such that, for each $r > 0$, the area of the sheets of R covering the circle $|w| < r$ does not exceed $p\pi r^2$. We shall call such functions mean p-valent, and the term mean univalent shall indicate the special case $p = 1$. We shall restrict our discussion to this latter case, since each p-valent function $f(z)$ can be replaced by the corresponding mean univalent function $f(z)^{1/p}$. Writing $w = \rho e^{i\varphi}$ in polar form, we can express the condition that $f(z)$ be mean univalent by requiring

$$\int_0^r \left\{ \int d\varphi - 2\pi \right\} \rho \, d\rho \leq 0 \tag{2}$$

for every $r > 0$, where it is understood that we integrate with respect to φ over all sheets of the Riemann surface R which project onto the circle $|w| = \rho$.

Spencer [5] has shown that within the class of mean univalent functions, the second power series coefficient a_2 satisfies the sharp inequality

(3) $$|a_2| \leq 2 \quad .$$

Spencer [5] also shows that if d is any value which f(z) does not assume in the unit circle, then $|d| > 1/7$, and he conjectures that the sharp estimate

(4) $$|d| \geq 1/4$$

is valid. The object of this note is to prove his conjecture.

To be precise, we consider analytic functions $w = f(z)$ of the form (1) in the unit circle, with $p = 1$, which map the unit circle onto a Riemann surface R over the w -plane satisfying the condition

(5) $$\int_0^r \left\{ \int d\varphi - 2\pi \right\} \frac{1}{\rho}\, d\rho \leq 0$$

for each $r > 0$, where the integration with respect to φ is extended over all sheets of R covering the circle $|w| = \rho$. For this class, which we shall call the class of <u>weakly mean univalent</u> functions, we show that any omitted value d satisfies the sharp inequality (4). Spencer's conjecture follows from this result, since (5) is less restrictive than (2). Condition (5) is an estimate of the area over certain half-planes of the Riemann surface D upon which $t = \log f(z)$ maps the unit circle. We state the result in the form of a

<u>Theorem</u>: <u>If</u> $w = f(z)$ <u>is a weakly mean univalent function of the type</u> (1) <u>and if</u> $f(z) \neq d$ <u>for</u> $|z| < 1$, <u>then</u> $|d| \geq 1/4$. <u>The same conclusion holds when</u> $w = f(z)$ <u>is mean univalent</u>.

The first part of our proof of (4) is based on the work of Hayman [3], who has given elegant sharp estimates for the distortion of p -valent mappings by using the concept of circular symmetrization due to Polya [4]. The later part of the proof depends on polygonal Hadamard variations [2] and closes with an inequality borrowed from the theory of free streamline flows.

2. Symmetrization and the Lindelöf principle.

The harmonic function

$$(6) \qquad \begin{aligned} G &= -\log|z| \\ &= -\log|w| + \gamma + O(|w|) \end{aligned}$$

is the Green's function of the Riemann surface R with pole at the origin. We call the constant γ the capacity of R at the origin. The normalization (1) of the mapping $w = f(z)$ is equivalent to the condition

$$(7) \qquad \gamma = 0$$

on the capacity constant. Since for any real θ the function $e^{-i\theta}f(ze^{i\theta})$ is weakly mean univalent whenever $f(z)$ is, there can be no loss of generality in the assumption that the omitted value d of $f(z)$ with smallest modulus $|d|$ is positive. The Riemann surface R does not cover the point d in the w -plane. Our problem is to choose R so that for γ satisfying (7) the number d is a minimum. It is easier to discuss this problem in an equivalent formulation in which we fix d and maximize γ.

We slit the Riemann surface R along the positive real axis between 0 and d, and on the slit surface we consider the fixed branch of the function

$$(8) \qquad \varphi = \arg w$$

which has values between 0 and 2π near the origin. We denote by R^* the Riemann surface obtained from R by identifying all points outside the circle $|w| < d$ which lie over the same point in the w -plane and have equal values for φ. The Riemann surface R^* may be multiply-connected, but R can be imbedded in it. Hence by Lindelöf's principle the capacity constant γ^* for R^* at the origin is not smaller than γ. At the same time, R^* does not cover d. Furthermore, R^* has no branch-points and satisfies condition (5)

when R does, since it is obtained from R by identifications.

We perform a process of circular symmetrization upon R^* which is defined as follows. For each $r \geqslant d$, we replace the arcs of R^* which cover the circle $|w| = r$ by a single open arc of the same circle with the same total length, situated in such a way that it is bisected by the negative real axis. We call the circularly symmetrized Riemann surface so obtained R^{**}, and we note that it is again simply-connected and without branch-points. The capacity constant γ^{**} of R^{**} at the origin is not smaller than γ^*, as can be seen from the work of Pólya [4]. The proof of the inequality $\gamma^{**} \geqslant \gamma^*$ depends merely upon expressing the capacity constant as a Dirichlet integral in terms of the Green's function, symmetrizing the Green's function in an evident manner, and estimating the Dirichlet integral of the symmetrized function by means of Dirichlet's principle. Finally, d is not covered by R^{**}, and (5) is satisfied by R^{**}, since the same is true of R^*.

The Riemann surface R^{**} is simply-connected and does not cover d, and furthermore,

$$\gamma^{**} \geqslant \gamma \qquad (9)$$

with equality holding only when $R^{**} = R$. Hence the Riemann surface for which $d > 0$ is given and γ is a maximum will be found among those which are invariant under circular symmetrization. Therefore we consider in the competition only such Riemann surfaces. That such an extremal surface exists is a consequence of the theorem that the class of weakly mean univalent functions forms a normal family.

A remark which will be important in the following is that, if the image D of R in the plane of the complex variable $t = \log w$ has a rectilinear polygonal boundary with at most a given number of vertices, and if R is circularly

symmetric, then the above processes of identification and circular symmetrization may alter D, but they take its polygonal boundary into another polygonal boundary with no more vertices than before.

3. Method of polygonal variations.

We wish to apply variational methods to the extremal problem posed in the previous section of maximizing the capacity γ of R for fixed $d > 0$. For this purpose we introduce a subclass $\Omega_{n,m}$ of the weakly mean univalent functions characterized by certain restrictions on the Riemann surface D in the t -plane. The functions f of class $\Omega_{n,m}$ map the unit circle $|z| < 1$ onto Riemann surfaces R over the w -plane which do not cover the point d and which are circularly symmetric in the sense that $f(z)$ is negative for negative values of z. The corresponding Riemann surfaces D in the logarithmic t -plane are assumed to be bounded by polygonal curves C with at most n vertices. Because of the multiple-valued nature of the mapping $t = \log w$ of R onto D, there are infinitely many congruent polygons C bounding D, but we fix our attention on that particular one which is symmetric in the real axis and passes through the real point $\delta = \log d$. We denote by C^+ and C^- the two symmetric branches of this polygon eminating from δ. We impose the final condition on the class $\Omega_{n,m}$ that the curve C^+ should rise from δ before falling and should cross the real axis at most m times, and that between two consecutive crossings t_k and t_{k+1} it should lie in the strip

$$t_k \leq \operatorname{Re}\{t\} \leq t_{k+1} \quad . \tag{10}$$

However, C^+ will be allowed to touch the real axis more than m times.

The class $\Omega_{n,m}$ is compact, and hence there is a function in $\Omega_{n,m}$ which, for a given $d > 0$, maximizes γ. We call the corresponding extremal Riemann surfaces in the w -plane and t -plane R and D, respectively. Let now

$P(t) = G + iH$ be an analytic function whose real part G is the Green's function of R with a logarithmic singularity at the origin. Hadamard's variational formula [1] shows that a shift of C by an amount εN along the inner normal of C which is congruent on all components of the boundary of D and which is symmetric on C^+ and C^- yields a shift

$$\delta\gamma = -\frac{\varepsilon}{\pi}\int_{C^+} |P'(t)|^2 N|dt| + o(\varepsilon) \tag{11}$$

of the capacity constant γ. This variational formula is easily derived for the present case of rectilinear boundary curves. From the extremal property of D, we find that when N represents a translation and rotation of each segment of C^+ which preserves the conditions (5) and (10), then $\delta\gamma \leq 0$.

Because of the circular symmetrization procedure of Section 2, we see that the extremal polygon C^+ in the logarithmic t-plane has the property that every vertical line intersects it in at most one segment or point. Let ℓ_ν denote the edges of C^+, ordered as we proceed from left to right with increasing $\operatorname{Re}\{t\}$. Our objective is to prove (4) by showing that C^+ actually reduces to one infinite segment of the real axis, but we start by assuming that this is not the case.

Suppose that ℓ_μ is a segment of C^+ lying above the real axis and that ℓ_ν is a later segment of C^+ lying below the real axis; suppose that neither of these segments lies on a vertical line through a point where C^+ crosses the real axis; and suppose that ℓ_μ and ℓ_ν are separated by only one point t_k where C^+ crosses the real axis. Then we can translate ℓ_μ and ℓ_ν by small amounts without altering (5) and (10). Such translations correspond to a normal shift N which is constant on ℓ_μ and on ℓ_ν. Thus the extremal property $\delta\gamma \leq 0$ of D gives by (11)

(12) $$\int_{\ell_\mu+\ell_\nu} |P'|^2 N|dt| \geqslant 0 \quad ,$$

whenever the area condition (5) remains unchanged, i.e., when the restriction

(13) $$\int_{\ell_\mu+\ell_\nu} N|dt| = 0$$

is fulfilled. Since (13) implies (12), we deduce that there exists a Lagrange multiplier λ_k such that

(14) $$\int_{\ell_\nu} |P'|^2|dt| = \lambda_k \int_{\ell_\nu} |dt|$$

when ℓ_ν is any segment of C^+ in the strip $t_{k-1} < \mathrm{Re}\{t\} < t_{k+1}$ which does not coincide with the real axis or the vertical line through t_k.

If two or more crossing points t_k separate ℓ_μ and ℓ_ν, then the constant normal displacement N on the first segment ℓ_μ must be positive in order to satisfy (5). Thus (14) cannot be obtained in this case, but we do find that

(15) $$\lambda_k \geqslant \lambda_{k+1} \quad .$$

As $n \to \infty$, the extremal functions of class $\Omega_{n,m}$ approach an extremal function which we shall term of class Ω_m. In the limiting case, the extremal Riemann surface D need not be bounded by polygons, and, indeed, we can prove that C^+ contains no rectilinear segment bordered by interior points of D and distinct from the real axis and the lines $\mathrm{Re}\{t\} = t_k$. For if there were such a segment on the boundary of D, we could apply (11) with an arbitrary normal displacement N and show that (13) implies (12). This would show, in turn, that $P'(t)^2$ is constant on that segment and hence constant throughout the t -plane, a manifest contradiction.

It follows from the symmetrized form of D and these remarks that as $n \to \infty$, an edge of C^+ either shrinks to a point, or approaches the real axis, or tends toward a vertical position. We shall prove that, away from the vertical lines

through the crossing points t_k, no segment of C^+ approaches a vertical segment.

Suppose, indeed, that a segment ℓ_ν of C^+ approached a vertical segment ℓ in the interior of the strip (10) as $n \to \infty$, and suppose, without loss of generality, that D bordered ℓ_ν on the right. Then the upper end-point of ℓ_ν tends to the upper end-point of ℓ, and some point T of C^+ to the right of ℓ_ν must also approach the upper end-point of ℓ. For if this were not the case, then the upper end of ℓ would be bordered by the Riemann surface D in the limiting case, and this would leave us in the previous contradictory situation.

Let ω be the function which is 0 on ℓ_ν, 0 on the vertical ray descending from ℓ_ν, 0 on the vertical ray descending from T, and 1 on the segment joining T to the upper end-point of ℓ_ν, and which is harmonic in the narrow region bounded by these lines. By the maximum principle, there is a positive constant M such that in that region $G \leq M\omega$, and hence $|P'(t)|^2 \leq M^2(\nabla\omega)^2$ on ℓ_ν. It follows that $P'(t) \to 0$ uniformly on ℓ_ν a fixed distance below the upper end-point of ℓ_ν, since $(\nabla\omega)^2 \to 0$ uniformly there.

Let s be arc length along ℓ_ν, measured from its center. Then infinitesimal rotation of ℓ_ν about its center shows by (11), with $N = s$, that

$$(16) \qquad \int_{\ell_\nu} |P'|^2 \, s\,ds = 0 \qquad ,$$

whereas (14) implies

$$(17) \qquad \underline{\lim} \int_{\ell_\nu} |P'|^2 \, ds > 0$$

as $n \to \infty$. Thus the center of gravity of the mass distribution with density $|P'|^2$ on ℓ_ν is located at the center of ℓ_ν, and the total mass remains above a certain positive bound. This contradicts the previous statement that $P'(t) \to 0$ uniformly a fixed distance below the upper end-point of ℓ_ν, and we conclude that ℓ_ν could not actually approach a vertical segment ℓ.

We introduce for each n and m the analytic function

(18) $$F(t) = -\int P'(t)^2 dt \quad ,$$

and we note that it maps any segment of the polygon C^+ onto another segment whose angle of inclination with the real axis is precisely the negative of the angle of inclination of the original segment. This follows from the fact that $\arg\,(P'(t))^2 dt = \arg\,(-d\bar{t})$ on a segment of C^+. Now (14) shows that the length of ℓ_ν and the length of its image by F are in the ratio λ_k. Thus F maps any arc C_k of C^+ which lies in a suitable strip $t_k + \eta \leq \mathrm{Re}\{t\} \leq t_{k+1} - \eta$, or $t_{k-1} + \eta \leq \mathrm{Re}\{t\} \leq t_k - \eta$, with $\eta > 0$, onto its reflection, stretched by the factor λ_k. Since $\lambda_k \bar{t}$ has the same property, we find that on C_k

(19) $$\left|\mathrm{Im}\left\{F(t) - \lambda_k \bar{t}\right\}\right| \leq \varepsilon_n \quad ,$$

where the number ε_n is the length of the longest projection on the imaginary axis of a segment ℓ_ν of C_k, and hence approaches zero as $n \to \infty$. The estimate (19) is even valid along segments of C^+ lying on the real axis, since these generate an error involving only the real parts of F and t in the above discussion.

We can now use the estimate (19) in the passage to the limit as $n \to \infty$. Using the Poisson integral representation of the harmonic function $\mathrm{Im}\left\{F(t) + \lambda_k t\right\}$ in the z -plane, we verify that for the Riemann surface D corresponding to a limiting weakly mean univalent function f of class Ω_m, we have

(20) $$\mathrm{Im}\left\{F(t) + \lambda_k t\right\} = 0$$

on boundary arcs C_k of C^+ between points where C^+ crosses the real axis. Thus $F + \lambda_k t$ can be reflected across the corresponding arc of the unit circle in the z -plane and is analytic on that arc as a function of z. Differentiation with respect to z shows that

$$(21) \qquad \left\{ \lambda_k - \frac{1}{z^2(dt/dz)^2} \right\} \frac{dt}{dz} = \left\{ \lambda_k - \frac{w^2/z^2}{dw^2/dz^2} \right\} \frac{1}{w} \frac{dw}{dz}$$

is an analytic function of z on this arc, and hence C_k consists of a finite number of analytic curves joined together by singularities of known type.

Thus far we have shown that extremal functions of class Ω_m generate extremal curves C^+ which are piece-wise analytic in the interiors of the strips (10) and satisfy (20) there. It is still conceivable, however, that C^+ contains vertical segments through the crossing points t_k. In order to discuss the nature of the singularities of C^+ at the end-points of such vertical portions, we notice that because the lengths of the sloping segments of approximating polygons of class $\Omega_{n,m}$ tend to zero as $n \rightarrow \infty$, the variational identity (14) and the fact that F(t) and t are imaginary on the vertical segment imply

$$(22) \qquad \operatorname{Re}\left\{ F(t) - \lambda_k t \right\} = 0$$

on C^+ in the neighborhood of one of these end-points. By (22), we can reflect $\operatorname{Re}\left\{ F - \lambda_k t \right\}$ across a corresponding arc of the unit circle in the z -plane and prove that C^+ has a singularity such that its tangent turns continuously.

Next we can show that no vertical segments of this type occur. For near a point of a vertical segment through t_k we can make a variation of C^+ by adding to or subtracting from D an infinitesimal, narrow rectangle. This may take us out of the class of admissible curves C^+, but if we then perform a circular symmetrization of the corresponding varied Riemann surface R in the w -plane, we do not decrease γ, and we do obtain an admissible C^+. Indeed, the new curve is montonoically ascending near t_k, since the old one had a continuously turning tangent, and hence it crosses the real axis at most once near t_k. Our rectangular variation is therefore permissible, and we can

use it, with the formula (11), to prove that $|P'(t)|^2$ is constant on the segment through t_k. But it follows that $P'(t)$ is then constant throughout D, and this contradiction implies that no vertical segment through t_k could have existed in the first place.

We conclude that (20) and (22) hold at all points of C^+ distinct from the real axis. Hence at such points of C^+

$$F(t) = \lambda_k \bar{t} \qquad (23)$$

whence

$$P'(t) = -i\lambda_k^{1/2}\,\bar{\dot{t}} \qquad (24)$$

by (18), where $\dot{t}$ is the unit tangent vector for C^+. Thus C^+ consists, in the limiting case of weakly mean univalent functions of class Ω_m, of a finite number of analytic arcs, some of which satisfy the free boundary condition (24), and the rest of which coincide with the real axis. The tangent of C^+ turns continuously and ends in a horizontal position at the point $\delta = \log d$. The last statement is proved by making a variation which consists in slitting R a short distance in from α toward the origin and then magnifying by a suitable factor. Unless C^+ has a horizontal tangent at δ, this variation increases γ. Finally, it is easy to see from (24) that sufficiently far to the right from the imaginary axis, C^+ reduces to an infinite ray of the real axis.

4. Estimate of the number of free boundaries.

In an effort to prove the inequality (4), we have applied variational analysis to the problem of maximizing the capacity γ for a fixed positive value of the omitted value d, in the class Ω_m. We have shown that the curve C^+ generated by the solution of this extremal problem consists of a finite number of analytic arcs satisfying the free boundary condition (24) and a

finite number of intervals of the real axis. We proceed to prove that actually no free boundary curves (24) occur at all, by making a suitable application of the argument principle. The idea of the proof is borrowed from the theory of free streamline flows, and is suggested by an interpretation of (24) as a condition for constant pressure.

We denote by D' the subregion of D bounded by C^+, by the interval $t < \delta$, and by the horizontal line

$$\operatorname{Im}\{t\} = \pi \tag{25}$$

The analytic function $P'(t)$ is regular in D' and has singularities on the boundary at the point at infinity, both to the right and to the left, at the point δ, and at the end-points of each free arc of C^+ satisfying (24). We exclude these singularities by small circles, or by segments, in the case of the point at infinity, and we calculate around the boundary of D' the integral

$$\Delta \arg P'' = \frac{1}{i} \int \frac{P'''(t)dt}{P''(t)} = 0 \tag{26}$$

We must also exclude by small circles any possible zeros of P''.

The point at infinity to the left contributes not more than $-\pi$ to $\Delta \arg P''$, since P'' behaves there like a power of e^t. The point at infinity to the right contributes $-\pi/2$, since P'' behaves like $e^{-t/2}$ there. The point δ contributes $3\pi/2$, since P'' has the behavior $(t-\delta)^{-3/2}$ there. The lines (25), $t < \delta$ and segments of the real axis to the right contribute a non-positive quantity to $\Delta \arg P''$, since P'' is regular and either real or pure imaginary there. Each point of separation α of a free arc of C^+ satisfying (24) contributes not more than $\pi/2$ because P'' behaves like $(t-\alpha)^{-\frac{1}{2}+q}$ there with $q \geqslant 0$. On the other hand, P'' vanishes at a point of inflection of C^+, which therefore contributes not more than $-\pi$. By Rolle's theorem and the

continuity of the tangent of C^+, each free arc of C^+ must have at least two points of inflection. Also, the remainder of such an arc contributes nothing to $\Delta \arg P''$, since by (24)

$$P'' = -\lambda_k \kappa \dot{\bar{t}}^2 \tag{27}$$

there, where κ is the curvature. Hence the total contribution of each free arc of C^+ does not exceed

$$\frac{\pi}{2} + \frac{\pi}{2} - \pi - \pi = -\pi \quad .$$

We conclude that if μ is the number of free arcs of C^+ satisfying (24), then

$$-\pi - \pi/2 + 3\pi/2 - \pi\mu \geq \Delta \arg P'' = 0 \quad . \tag{28}$$

It follows from this inequality that $\mu \leq 0$, which proves that C^+ contains no free arcs and must reduce to the interval $t \geq \delta$.

An extremal function f which maximizes γ for given δ without restrictions other than (5) can be approximated arbitrarily by functions of class $\Omega_{n,m}$. Among these functions, the largest value of γ is given by that function f_0 which maps $|z| < 1$ on the entire plane slit along the positive real axis from d to $+\infty$, and hence f_0 must possess the largest value of γ among all weakly mean univalent functions. If we take $d = 1/4$, we check immediately that f_0 is the Koebe function

$$f_0(z) = \frac{z}{(1-z)^2} = \sum_{k=1}^{\infty} kz^k \quad , \tag{29}$$

for which $\gamma = 0$. The inequality (4) for weakly mean univalent functions, and therefore also for mean univalent functions in the sense of Spencer, is a direct consequence of this conclusion.

The reader will see for himself that our method leads to many further interesting results for the class of mean univalent functions.

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Chicago 37, Illinois 1

Los Angeles Engineering Field Office
Air Research and Development Command
5504 Hollywood Boulevard
Los Angeles 28, California
Attn: Capt. N. E. Nelson 1

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Navy Department Naval Ordnance Test Station Underwater Ordnance Department Pasadena, California Attn: Dr. G. V. Schliestett Code P8001	1
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